An Infinity of Infinities!

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1 Introduction

Infinity. I am sure we all have heard of the word at some moment in our lives. 'The Universe is infinite' or 'Time is infinite'. But what does it mean for something to be infinite? At the surface, it looks simple enough. In common terminology, it means unending, or incessantly vast. As a number, infinity is simply a number larger than any possible natural number. That is a definition sufficient for most laypersons, and even most scientists. However, mathematicians are wily creatures who are not quite satisfied with simple things in life. A guy called Georg Cantor came up one day and asked himself, "What exactly is this infinity business? How much really is infinity? And is there anything bigger than infinity? And are all infinities the same?" Weird questions, you might say. But considering that the fellow was Georg Cantor, and he became famous enough for us to know his name today, the questions weren't that foolish after all.

In this article, we shall present in some amount of mathematical rigour, the notions of infinity. Many of these might seem counter-intuituve to you. In fact, you might also ask why all of this should be necessary. But in mathematics, definitions need to be strict, so that we do not have inconsistencies and paradoxes. For practical purposes, the layman's notions could be sufficient, but not to a mathematician who is trying to consolidate the foundations of mathematics.

2 Sets and Sizes of 'Infinities'

Infinity is best defined in terms of sets. A set is simply a collection of things (things which may themselves also be sets). Sets may be infinite in size. For example, the set of all natural numbers is a set {1,2,3,4,...} which is infinite in size. The set of integers is also infinite in size {...-3,-2,-1,0,1,2,3......}. The question is whether these two infinities are the 'same in size', whatever that means. At first look it looks like they are not. Clearly, the set of integers has 'more things' than the set of naturals. Everything that is present in the set of naturals is present in the set of integers, plus more. However - and here is a very critical argument - if we are able to establish a one-to-one mapping from one set to the other, we say that the two sets are equal in size. For example {0,1,2,3} and {2,3,4,5} have the same size, since we can define a mapping

- $0 \to 2$
- $1 \to 3$
- $2 \to 4$
- $3 \to 5$

The same must be true for infinite sets as well, right? And indeed, in case of natural numbers and integers, we can indeed make such a mapping

- $1 \to 0$
- $2 \to 1$
- $3 \to -1$
- $4 \to 2$
- $5 \to -2$
- ....and so on

Note that we would never 'run out' of either natural numbers or integers. Also note that each natural number and each integer, will eventually be covered in this mapping. Hence, this seems like a legitimate mapping - indeed it is - and thus we are forced to admit that the size of the set of integers is same as the size of the set of natural numbers. This is slightly puzzling and counterintuitive, but true. You will never see such puzzling things happening with finite sets. But the human mind cannot see infinite sets in their entirety, and hence it is not possible to reason about them so trivially. However, on some other level, it does not seem so surprising that integers are the same in number as naturals, since the set of integers is roughly twice as large as the set of naturals, and $2 \times \infty = \infty$ seems kind of okay. However, is $\infty \times \infty = \infty$? Let us construct a set whose size is $\infty \times \infty$.

Consider the set of all ordered pairs of natural numbers. This set would contain things like (1,1), (1,2), (2,1), (4534,4), (9421066285,5318008) etc. It is an infinite set, but is this set also the same in size as the set of natural numbers? Let us try the technique we used above - that of making a mapping to natural numbers. Let's try such a mapping

- $(1,1) \to 1$
- $(1,2) \to 2$
- $(1,3) \to 3$
- $(1,4) \to 4$

- ....and so on

The problem in this mapping is clearly apparent. We would keep going to larger and larger natural numbers, but the (1,x) pairs would never end. (2,1) would map to nothing! Then is all hope lost? Is it indeed so that this set is 'larger' than the set of naturals? Hold on! We can define another mapping:

- $(1,1) \to 1$
- $(2,1) \to 3$
- $(1,2) \to 2$
- $(3,1) \to 6$
- $(2,2) \to 5$
- $(1,3) \to 4$
- ....and so on

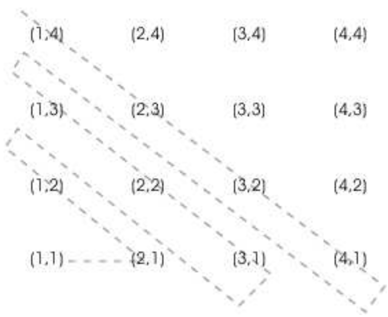

Figure 1: Mapping for pairs of Naturals

The figure will help you clearly visualize this mapping. Once again, as in the case of integers, we will run out of neither pairs nor naturals, and all pairs and all naturals will eventually be covered in the mapping. Thus, the size of the set of pairs is also the same as the size of the set of natural numbers.

3 A bigger Infinity

This discussion might tempt you to conclude that 'all infinities are equal in size'. But our beloved Georg Cantor proved that this is not the case. Consider subsets of natural numbers. These may be finite subsets like {1,2,3}, {4,5,6,324987} and also infinite subsets like {1,3,5,7,.....}, {1,11,111,1111.....} and {1,2,3,4.....} itself. Now consider the set of all such subsets. Call this set the BAAP of natural numbers. Clearly, BAAP has infinitely many things in it. So can we compare the infinity that is the size of BAAP, to the infinity that is the size of the set of natural numbers? Is this also the same? Georgie proved that this is not the case. He did not merely fail to come up with a suitable mapping. He proved that nobody can come up with a suitable mapping. The natural numbers would run out before all the elements of BAAP are covered. Thus, he proved that BAAP contains strictly more elements than the set of natural numbers. We can understand this by observing that number of elements in BAAP will be 2^{∞}. This is exponentially greater than the number of elements in the set of naturals, and it turns out that this is just too much for the set of naturals to 'expand into'.

4 Conclusion

In fact, Georgie went one step further. He proved that there are an infinite number of infinities, each strictly greater than the previous one, and the size of the set of these infinities is the same as the size of the set of natural numbers. If you find all this weird and hard to digest, don't worry. You are not alone. A lot of Georg Cantor's contemporaries scoffed at his theories, and he faced opposition from fellow mathematicians and even the church. However, Cantor's theories are well established and accepted universally now. Though he faced ostracision and ridicule in his life, he received great posthumous glory, and his name is remembered today as one of the greats in mathematics.